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On the First and Second Logarithmic Derivatives of Hyperelliptic σ-Functions.

By Oskar Bolza.

Introduction.

Let

$$R(x) = A_0x^4 + 4A_1x^3 + 6A_2x^2 + 4A_3x + A_4$$

be a quartic whose discriminant is different from zero; further,

$$y^2 = R(x), \quad \eta^2 = R(\xi)$$

and

$$w^{x\xi} = \int_{(\xi,\, \eta)}^{(x,\, y)} \frac{dx}{y};$$

then, according to well-known theorems of the theory of elliptic functions,[*]

$$\frac{\sigma'}{\sigma}(w^{x\xi}) = \frac{1}{2}\,\frac{y+\eta}{x-\xi} - \int_{(\xi,\, \eta)}^{(x,\, y)} (A_0x^2 + 2A_1x + A_2)\frac{dx}{2y} \tag{a}$$

and[†]

$$\wp(w^{x\xi}) = \frac{y\eta + F(x, \xi)}{2(x-\xi)^2}; \tag{b}$$

$F(x, \xi)$ being the second polar[‡] of ξ with regard to $R(x)$. In particular, a being a root of $R(x)$,[||]

$$\wp(w^{xa}) = \frac{1}{24}R''(a) + \frac{1}{4}\,\frac{R'(a)}{x-a}, \tag{c}$$

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[*] I cannot refer to a place where (a) is given precisely in this form; it follows, however, readily from a formula proved by Weierstrass in one of his courses on Elliptic Functions, viz. If $u = w^{xa}$, $u_0 = w^{\infty a}$, a being a branchpoint, then $\wp(u + u_0) + \wp(u - u_0) = A_0x^2 + 2A_1x + A_2$, or else it may be derived from (b).

[†] Weierstrass, Lectures on Elliptic Functions. See Enneper-Müller, Elliptische Functionen, p. 29. The covariant character of the formula was first recognized by Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 27, p. 464; see also Halphen, Traité des fonctions elliptiques, II, p. 359, and Harkness and Morley, Theory of Functions, p. 340.

[‡] That is, if we introduce homogeneous variables, $x = x_1/x_2$, $\xi = \xi_1/\xi_2$ and put $x_2^4 R(x) = a_x^4$, then $F(x, \xi)\, \xi_2^2 x_2^2 = a_x^2\, a_\xi^2$.

[||] Weierstrass, Lectures on Elliptic Functions; see Enneper-Müller, l. c., p. 30.

and differentiating

$$\wp'(w^{xa}) = -\frac{1}{4}\,\frac{R'(a)}{(x-a)^2}\,y\,. \tag{d}$$

If a' be another root of $R(x)$ and

$$\phi(x) = (x-a)(x-a'), \quad R(x) = \phi(x)\,\psi(x),$$

then*

$$\wp(w^{aa'}) = -\frac{1}{6}\,(\phi\psi)^2, \tag{e}$$

$(\phi\psi)^2$ denoting the second transvectant of

$$\phi_x^2 = x_2^2\phi\left(\frac{x_1}{x_2}\right) \text{ and } \psi_x^2 = x_2^2\psi\left(\frac{x_1}{x_2}\right).$$

In the following paper I propose to give an *extension of these and some allied theorems to hyperelliptic functions.*

PART I.

THE FIRST LOGARITHMIC DERIVATIVES.

§1.—*Notations.*

Let

$$y^2 = R(x) \equiv \sum_{i=0}^{2\rho+2}\binom{2\rho+2}{i} A_i x^{2\rho+2-i} \equiv A_0 \prod_{k=0}^{2\rho+} (x-a_k) \tag{1}$$

be the *hyperelliptic curve* under consideration, of deficiency ρ; further,

$$w_\alpha^{x\xi} = \int_{(\xi,\eta)}^{(x,y)} \frac{g_\alpha(x)\,dx}{y}, \qquad (\alpha = 1,\,2,\,\ldots\,\rho) \tag{2}$$

a set of ρ linearly independent *integrals of the first kind,* the $g_\alpha(x)$'s being integral functions of degree not higher than the $(\rho-1)^{\text{st}}$; and

$$Q_{\xi_1\xi_0}^{x_1x_0} = \int_{(x_0,y_0)}^{(x_1,y_1)} \int_{(\xi_0,\eta_0)}^{(\xi_1,\eta_1)} \frac{y\eta + F(x,\xi)}{2(x-\xi)^2\,y\eta}\,dx\,d\xi, \tag{3}$$

Klein's commutative *integral of the third kind,*† $F(x,\xi)$ being the $(\rho+1)^{\text{st}}$ polar of ξ with regard to $R(x)$, in the same sense as above.

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* Klein, Lectures on Hyperelliptic Functions, 1887. In Math. Ann., Bd. 27, p. 459, (64), the factor $\frac{1}{3}$ should be changed into $-\frac{1}{6}$.

† Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 32, p. 365. I use, however, non-homogeneous notation, which is better adapted for our present purpose.

The "*normal integrals of the second kind*":[*]

$$w^{x\xi}_{\rho+\alpha}=\int_{(\xi,\,\eta)}^{(x,\,y)}\frac{g_{\rho+\alpha}(x)\,dx}{y}\,,\qquad(\alpha=1,\,2,\,\ldots.\rho)\tag{4}$$

associated with the integrals (2) and (3), are determined by the identical relation[†]

$$\frac{d}{d\xi}\,\frac{1}{2}\,\frac{y+\eta}{(x-\xi)\,y}-\sum_{\alpha}\frac{g_{\alpha}(x)}{y}\,\frac{g_{\rho+\alpha}(\xi)}{\eta}=\frac{y\eta+F(x,\,\xi)}{2\,(x-\xi)^2\,y\eta}\tag{5}$$

which furnishes for the $g_{\rho+\alpha}(x)$'s integral functions each of degree not higher than the $2\rho^{\text{th}}$.[‡]

With a system of canonical periods

$$\begin{matrix}2\omega_{\alpha\beta}, & 2\omega'_{\alpha\beta}, \\ 2\eta_{\alpha\beta}, & 2\eta'_{\alpha\beta},\end{matrix}\tag{6}$$

of the integrals (2) and (4) as "moduli," we construct the $2^{2\rho}$ Θ-*functions*[§] of ρ independent variables $u_1,\,u_2,\,\ldots.\,u_\rho$. They differ only by constant factors—which are of no consequence for our present purpose—from Klein's σ-*functions*:[||]

$$\sigma_{\phi\psi}(u_1,\,u_2,\,\ldots.\,u_\rho)=\frac{\Theta_{\left[\substack{g\\h}\right]}(u_1,\,u_2,\,\ldots.\,u_\rho)}{C_{\phi\psi}}\,.\tag{7}$$

The correspondence between[**] the "transcendental characteristics" $\begin{bmatrix}g\\h\end{bmatrix}$ and the "algebraic characteristics" $\phi\psi$ depends on the canonical dissection of the Riemann-surface. In order to fix the ideas we suppose the dissection so chosen that the algebraic characteristic $\phi_0\psi_0$ of the "fundamental" Θ-function $\Theta_{\left[\begin{smallmatrix}0 \dots\dots 0\\ 0 \dots\dots 0\end{smallmatrix}\right]}$ is

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[*] Weierstrass, Lectures on Hyperelliptic Functions. See Wiltheiss, Journal für Math., Bd. 99, p. 238; Math. Ann., Bd. 31, p. 136, and Bd. 33, p. 269. I follow here the notation used by Weierstrass in his course of winter 1881-82, which differs slightly from Wiltheiss' notation; accordingly I write $g_\alpha(x)$ and $g_{\rho+\alpha}(x)$ instead of Wiltheiss' $\frac{H_\alpha(x)}{2}$ and $\frac{G_\alpha(x)}{2}$.

[†] As in Weierstrass' lectures, *the letters α, β, γ will always be used to denote indices running from 1 to ρ.*

[‡] The explicit expressions of these integral functions are of no consequence for our investigation; they may, however, be found in my paper "On Weierstrass' systems of hyperelliptic integrals of the first and second kind," read before the Mathematical Congress in Chicago, 1893.

[§] Weierstrass, Lectures; see Schottky, Abel'sche Functionen, §1, and Wiltheiss, ll. cc.

[||] Klein, Math. Ann., Bd. 32, pp. 367, 376; Burkhardt, Math. Ann., Bd. 32, pp. 418, 427; and Schröder, Ueber den Zusammenhang der hyperelliptischen σ- und ϑ-Functionen, Diss. Göttingen, 1890.

[**] See Prym, Functionen in einer zweiblättrigen Fläche, Zürich, 1866, Art. 6; Klein, Lectures on Hyperelliptic Functions, 1887-88, and Thompson, Hyperelliptische Schnittsysteme und Zusammenordnung der algebraischen und transcendentalen Thetacharacteristiken, *American Jour. of Math.*, Vol. XV, p. 91.

$$\begin{aligned}\phi_0(x) &= \text{const.}\,(x-a_1)(x-a_3)\dots\dots(x-a_{2\rho+1}),\\ \psi_0(x) &= \text{const.}\,(x-a_0)(x-a_2)\dots\dots(x-a_{2\rho}).\end{aligned}\tag{8}$$

After these preliminaries we *define*

$$\zeta_{\phi\psi}(u_1, u_2, \dots\dots u_\rho;\, t) = \sum_\alpha \frac{\partial \log \sigma_{\phi\psi}(u_1, u_2, \dots\dots u_\rho)}{\partial u_\alpha}\, g_\alpha(t),\tag{9}$$

t being an auxiliary independent variable. The letter ζ is chosen to indicate the analogy with the elliptic function $\zeta u = \frac{\sigma'}{\sigma}(u)$. In fact, for $\rho = 1$ and $g_1(x) = 1$, we obtain the four functions

$$\frac{\sigma'}{\sigma}(u),\ \frac{\sigma_1'}{\sigma_1}(u),\ \frac{\sigma_2'}{\sigma_2}(u),\ \frac{\sigma_3'}{\sigma_3}(u).$$

The introduction of the auxiliary variable t—which may at first sight seem rather artificial—allows to unite in a very convenient way theorems concerning the *system of ρ functions,*

$$\frac{\partial \log \sigma}{\partial u_1},\ \frac{\partial \log \sigma}{\partial u_2},\ \dots\dots \frac{\partial \log \sigma}{\partial u_\rho}.$$

A last remark concerning notations refers to homogeneous variables. For the greater part of our work it is simpler to use non-homogeneous variables, but sometimes homogeneous variables are preferable. In order to avoid a complication of notations, the same functional symbol $f(x)$ will be used to designate an integral *function* of x, say of degree n, when non-homogeneous variables are used, and the binary *quantic* $x_2^n f\left(\frac{x_1}{x_2}\right)$ when homogeneous variables are used. Thus $\zeta(u_1, \dots\dots u_\rho;\, t)$ will occasionally stand for the homogeneous function of t_1, t_2:

$$t_2^{\rho-1}\zeta\left(u_1, \dots\dots u_\rho;\, \frac{t_1}{t_2}\right).$$

§2.—*Expression of ζ in terms of the Normal Integrals of the Second Kind.*

We now *replace in $\zeta(u_1, u_2, \dots\dots u_\rho;\, t)$* the independent variables u_α by sums of $\nu \overset{=}{>} \rho$ integrals of the first kind,*

$$w_\alpha = w_a^{x_1\xi_1} + w_a^{x_2\xi_2} + \dots\dots w_a^{x_\nu\xi_\nu}.\tag{10}$$

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* We suppress the characteristic $\phi\psi$ where no distinction between different characteristics is necessary.

Our first problem is to express $\zeta(w_1, \ldots . w_\rho; t)$ in terms of the normal integrals of the second kind, $w^{x_i \xi_i}_{\rho + \alpha}$.

We begin with the following

Lemma: If $\pi(t)$ denote the product

$$\pi(t) = \prod_i (t - x_i),$$

then

$$\zeta(w_1, \ldots . w_\rho; t) = \sum_i \frac{\pi(t)}{\pi'(x_i)(t - x_i)} y_i \frac{\partial \log \sigma}{\partial x_i}, \tag{11}$$

the index i running from 1 to ν.

Proof: Logarithmic differentiation of $\sigma(w_1, \ldots . w_\rho)$ with respect to x_i gives

$$\frac{\partial \log \sigma}{\partial x_i} = \sum_\alpha \frac{\partial \log \sigma}{\partial w_\alpha} \frac{g_\alpha(x_i)}{y_i},$$

or, according to (9):

$$y_i \frac{\partial \log \sigma}{\partial x_i} = \zeta(w_1, \ldots . w_\rho; x_i).$$

But since $\zeta(w_1, \ldots . w_\rho; t)$ is an integral function of t of degree $\rho - 1$, and $\nu \geqq \rho$, we have, by decomposition into partial fractions,

$$\frac{\zeta(w_1, \ldots . w_\rho; t)}{\pi(t)} = \sum_i \frac{\zeta(w_1, \ldots . w_\rho; x_i)}{\pi'(x_i)(t - x_i)} = \sum_i \frac{y_i \frac{\partial \log \sigma}{\partial x_i}}{\pi'(x_i)(t - x_i)}. \qquad \text{Q. E. D.}$$

Corollary: Evidently (11) remains true if in the sum $\sum\limits_i$ and in the product $\prod$ the index i takes—not all but—*at least ρ of the ν values* $1, 2, \ldots . \nu$. More generally, if $h(t)$ denote an arbitrary polynomial whose degree is $\leqq \nu - \rho$, then

$$\zeta(w_1, \ldots . w_\rho; t) = \sum_{i=1}^{\nu} \frac{\pi(t)\, h(x_i)}{\pi'(x_i)(t - x_i)\, h(t)} y_i \frac{\partial \log \sigma}{\partial x_i}. \tag{12}$$

Our problem is now reduced to the computation of $\frac{\partial \log \sigma}{\partial x_i}$; to this effect we start from Klein's explicit expression of $\sigma_{\phi\psi}(w_1, \ldots . w_\rho)$ in terms of $x_1, \ldots . x_\nu; \xi_1, \ldots . \xi_\nu$:*

Let

$$\Omega(x, \xi) = \frac{(x - \xi)}{\sqrt{y\eta}} e^{\frac{1}{2} Q^{\bar{x}\bar{\xi}}_{x\xi}}, \tag{13}$$

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* Klein, Hyperelliptische σ-Functionen, Math. Ann., Bd. 32, p. 367.

$\bar{x}$ and $\bar{\xi}$ denoting the points $(x, -y)$ and $(\xi, -\eta)$ conjugate with the points $x = (x, y)$ and $\xi = (\xi, \eta)$.

Further, if $\rho + 1 - 2\mu$ and $\rho + 1 + 2\mu$ are the respective degrees of ϕ and ψ, let $D_{\phi\psi}$ denote the determinant of order 2ν:

$$D_{\phi\psi} = \tag{14}$$

$$\left| \begin{array}{l} \sqrt{\phi(x_i)}, \quad x_i\sqrt{\phi(x_i)} \ldots\ldots \quad x_i^{\nu-1+\mu}\sqrt{\phi(x_i)}\,;\ \sqrt{\psi(x_i)},\ x_i\sqrt{\psi(x_i)} \ldots\ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots x_i^{\nu-1-\mu}\sqrt{\psi(x_i)} \\ -\sqrt{\phi(\xi_i)}, \quad -\xi_i\sqrt{\phi(\xi_i)} \ldots\ldots \quad -\xi_i^{\nu-1+\mu}\sqrt{\phi(\xi_i)}\,;\ \sqrt{\psi(\xi_i)},\ \xi_i\sqrt{\psi(\xi_i)} \ldots\ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots \xi_i^{\nu-1-\mu}\sqrt{\psi(\xi_i)}, \end{array} \right|$$

$$i = 1,\ 2,\ \ldots\ldots \nu,$$

then

$$\sigma_{\phi\psi}(w_1, \ldots\ldots w_\rho) = c_\nu^{(\mu)} \frac{D_{\phi\psi}\prod\limits_i\prod\limits_k \Omega(x_i, \xi_k)}{\prod\limits_i\prod\limits_k (x_i - \xi_k) \prod\limits_{(i,k)} \Omega(x_i, x_k) \prod\limits_{(i,k)} \Omega(\xi_i, \xi_k)}; \tag{15}$$

$c_\nu^{(\mu)}$ is a numerical constant which has no influence upon our conclusions; in the double products $\prod\limits_i\prod\limits_k$, i and k take independently the values $1, 2, \ldots\ldots \nu$, whereas the products $\prod\limits_{(i,k)}$ extend over the $\frac{\nu(\nu-1)}{2}$ two-combinations of the numbers $1, 2, \ldots\ldots \nu$.

By logarithmic differentiation of (15) we get

$$\frac{\partial \log \sigma}{\partial x_i} = \frac{\partial \log D}{\partial x_i} - \sum_k \frac{1}{x_i - \xi_k} + \sum_k \frac{\partial \log \Omega(x_i, \xi_k)}{\partial x_i} - {\sum_k}' \frac{\partial \log \Omega(x_i, x_k)}{\partial x_i}, \tag{16}$$

where $\sum\limits_k$ extends over all the values $1, 2, \ldots\ldots \nu$, ${\sum\limits_k}'$ over all the values $1, 2, \ldots\ldots \nu$ with the exception of $k = i$.

But from (13) and (3) follows

$$\frac{\partial \log \Omega(x_i, x_k)}{\partial x_i} = \frac{1}{x_i - x_k} - \frac{1}{4}\,\frac{R'(x_i)}{R(x_i)} + \frac{1}{2}\,\frac{\partial}{\partial x_i}\,Q_{x_i x_k}^{\bar{x}_i \bar{x}_k},$$

and

$$\frac{\partial Q_{x_i x_k}^{\bar{x}_i x_k}}{\partial x_i} = \int_{\bar{x}_k}^{\bar{x}_i} \frac{y_i\eta + F(x_i, \xi)}{2(x_i - \xi)^2 y_i\eta}\,d\xi + \int_{x_k}^{x_i} \frac{\bar{y}_i y + F(\bar{x}_i, x)}{2(\bar{x}_i - x)^2 \bar{y}_i y}\,dx,$$

and since $\bar{x}_i = x_i$, $\bar{y}_i = -y_i$,

$$\frac{\partial Q_{x_i x_k}^{\bar{x}_i \bar{x}_k}}{\partial x_i} = 2\int_{x_k}^{x_i} \frac{y_i y - F(x_i, x)}{2(x_i - x)^2 y_i y}\,dx.$$

But according to (5),

$$\frac{y_i y - F(x_i,\, x)}{2\,(x_i - x)^2\, y_i y} = \frac{d}{dx}\;\frac{1}{2}\;\frac{y - y_i}{(x - x_i)\, y_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\;\frac{g_{\rho+\alpha}(x)}{y}\,;$$

hence integrating,

$$\int_{x_k}^{x_i} \frac{y_i y - F(x_i,\, x)}{2\,(x_i - x)^2\, y_i y}\, dx = \Big[\frac{1}{2}\;\frac{y - y_i}{(x - x_i)\, y_i}\Big]_{x_k}^{x_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\, w_{\rho+\alpha}^{x_i x_k},$$

and since

$$\Big[\frac{1}{2}\;\frac{y - y_i}{(x - x_i)\, y_i}\Big]_{x = x_i} = \frac{1}{2}\;\frac{R'(x_i)}{R(x_i)}\,,$$

we get

$$\frac{\partial \log \Omega\,(x_i x_k)}{\partial x_i} = \frac{y_i + y_k}{2\,(x_i - x_k)\, y_i} + \sum_{\alpha} \frac{g_\alpha(x_i)}{y_i}\, w_{\rho+\alpha}^{x_i x}. \tag{17}$$

Returning to (16) and writing

$$w_{\rho+\alpha} = w_{\rho+\alpha}^{x_1 \xi_1} + w_{\rho+\alpha}^{x_2 \xi_2} + \dots + w_{\rho+\alpha}^{x_\nu \xi_\nu}, \tag{18}$$

we obtain

$$y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = y_i \frac{\partial \log D_{\phi\psi}}{\partial x_i} - \frac{1}{2}\sum_{k} \frac{y_i - \eta_k}{(x_i - \xi_k)} - \frac{1}{2}\sum_{k}{}' \frac{y_i + y_k}{x_i - x_k} + \sum_{\alpha} g_\alpha(x_i)\, w_{\rho+\alpha}. \tag{19}$$

Substituting this value in (11) and remembering that

$$\sum_{i} \frac{\pi(t)\, g_\alpha(x_i)}{\pi'(x_i)(t - x_i)} = g_\alpha(t),$$

we obtain the result:

Theorem: If

$$w_\alpha = w_\alpha^{x_1 \xi_1} + w_\alpha^{x_2 \xi_2} + \dots + w_\alpha^{x_\nu \xi_\nu},$$

and

$$w_{\rho+\alpha} = w_{\rho+\alpha}^{x_1 \xi_1} + w_{\rho+\alpha}^{x_2 \xi_2} + \dots + w_{\rho+\alpha}^{x_\nu \xi_\nu},$$

then

$$\zeta_{\phi\psi}(w_1 \dots w_\rho\,;\; t) = \sum_{\alpha} g_\alpha(t)\, w_{\rho+\alpha} + \sum_{i} \frac{\pi(t)}{\pi'(x_i)(t - x_i)} \Big\{ y_i \frac{\partial \log D_{\phi\psi}}{\partial x_i} - \frac{1}{2}\sum_{k} \frac{y_i - \eta_k}{x_i - \xi_k} - \frac{1}{2}\sum_{k}{}' \frac{y_i + y_k}{x_i - x_k} \Big\}. \tag{A}$$

§3.—*Invariantive Properties of* ζ.

The function $\zeta(w_1, \ldots . w_\rho;\, t)$ is a *covariant* with respect to a cogredient linear transformation of the variables $x_1, x_2, \ldots . x_\nu;\ \xi_1, \xi_2, \ldots . \xi_\nu;\ t$. In fact, apply to the hyperelliptic curve (1) the transformation

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta} \qquad y = \frac{y'}{(\gamma x' + \delta)^{\rho+1}} \tag{20}$$

$$\alpha\delta - \beta\gamma = r \ddagger 0,$$

which changes (1) into

$$y'^2 = R_1(x') \equiv (\gamma x' + \delta)^{2\rho+2} R(x). \tag{21}$$

According to a fundamental theorem due to Klein,[*] σ is a covariant with respect to this transformation; that is to say, let

$$(x_1', y_1'),\ (x_2', y_2'), \ldots . (x_\nu', y_1');\ (\xi_1', \eta_1'), \ldots . (\xi_\nu', \eta_\nu')$$

be the points into which the limits

$$(x_1 y_1),\ (x_2 y_2), \ldots . (x_\nu y_\nu);\ (\xi_1 \eta_1), \ldots . (\xi_\nu \eta_\nu)$$

are changed by the transformation (20); further,

$$\phi_1(x') = (\gamma x' + \delta)^{\rho+1-2\mu} \phi(x),$$
$$\psi_1(x') = (\gamma x' + \delta)^{\rho+1+2\mu} \psi(x);$$

then if we denote by $\sigma'_{\phi_1\psi_1}$ the σ-function derived from the curve (20), the limits x_i', ξ_i' and the characteristic $\phi_1\psi_1$ precisely in the same way as $\sigma_{\phi\psi}$ is derived from the curve (20), the limits x_i, ξ_i and the characteristic $\phi\psi$:[†]

$$\sigma_{\phi\psi} = r^{\mu^2} \sigma'_{\phi_1\psi_1}. \tag{22}$$

Hence by logarithmic differentiation,

$$y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = \frac{r^{-1}}{(\gamma x_i' + \delta)^{\rho-1}}\, y_i' \frac{\partial \log \sigma'_{\phi_1\psi_1}}{\partial x_i'},$$

and if we put

$$t = \frac{\alpha t' + \beta}{\gamma t' + \delta},$$

and

$$\pi_1(t') = \prod_i (t' - x_i'),$$

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[*] Math. Ann., Bd. 32, p. 370.

[†] Observe that the expression (15) of $\sigma_{\phi\psi}(w_1, \ldots w_\rho)$ in terms of $x_1, \ldots x_\nu;\ \xi_1, \ldots \xi_\nu$ is independent of the choice of the function $g_\alpha(x)$.

we obtain

$$\sum_i \frac{\pi(t)}{\pi'(x_i)(t-x_i)}\, y_i \frac{\partial \log \sigma_{\phi\psi}}{\partial x_i} = \frac{r^{-1}}{(\gamma t'+\delta)^{\rho-1}} \sum_i \frac{\pi_1(t')(\gamma x_i'+\delta)^{\nu-\rho}}{\pi_1'(x_i')(t'-x_i')(\gamma t+\delta)^{\nu-\rho}}\, \frac{y_i' \partial \log \sigma'_{\phi_1\psi_1}}{\partial x_i'} .$$

But according to (11) and (12) this is

$$\zeta_{\phi\psi}(w_1, \ldots . w_\rho;\, t) = \frac{r^{-1}}{(\gamma t'+\delta)^{\rho-1}} \zeta'_{\phi_1\psi_1}(w_1', \ldots . w_\rho';\, t'), \tag{23}$$

If, finally, we introduce homogeneous variables and employ the abbreviated notation agreed upon at the end of §1, equation (23) takes the simple form

$$\zeta_{\phi\psi}(w_1, \ldots . w_\rho;\, t) = r^{-1} \zeta'_{\phi_1\psi_1}(w_1', \ldots . w_\rho';\, t), \tag{24}$$

and we may therefore say:

The function $\zeta(w_1, \ldots . w_\rho;\, t)$ *is a covariant of* $R(x)$ *of weight* -1, *with the* $2\nu+1$ *sets of cogredient variables* $x_1 x_2 \ldots . x_\nu;\ \xi_1 \xi_2 \ldots . \xi_\nu;\ t$.

§4.—*Generalization of Theorem* (*a*).

Formula (A) simplifies considerably under the following two restricting assumptions:

In the first place we suppose $\mu = 0$, and in order to fix the ideas we confine ourselves to the "fundamental" σ-function whose algebraic characteristic $\phi_0\psi_0$ is given by (8); adopting a notation occasionally used by Professor Klein in his lectures, we designate it by σ_0.

In the second place we assume $\nu = \rho + 1$ and choose for the lower limits ξ_k the roots a_{2k-1} of ϕ_0, so that

$$w_a = w_a^{x_1 a_1} + w_a^{x_2 a_3} + \ldots . + w_a^{x_{\rho+1} a_{2\rho+1}}. \tag{25}$$

These assumptions reduce $D_{\phi\psi}$ to

$$D_0 = \text{const.} \prod_i \sqrt{\phi_0(x_i)} \prod_{(i,\,k)} (x_i - x_k);$$

hence

$$\frac{\partial \log D_0}{\partial x_i} = \frac{1}{2}\, \frac{\phi_0'(x_i)}{\phi_0(x_i)} + \sum_k{}' \frac{1}{x_i - x_k},$$

and

$$y_i \frac{\partial \log \sigma_0}{\partial x_i} = \frac{1}{2} \sum_k{}' \frac{y_i - y_k}{x_i - x_k} + \sum_a g_a(x_i)\, w_{\rho+a}. \tag{26}$$

Substituting this value in (A), we obtain the

Theorem: *If*

$$w_a = w_a^{x_1 a_1} + w_a^{x_2 a_3} + \ldots . + w_a^{x_{\rho+1} a_{2\rho+1}},$$

and

$$w_{\rho+a} = w_{\rho+a}^{x_1 a_1} + w_{\rho+a}^{x_2 a_3} + \ldots . + w_{\rho+a}^{x_{\rho+1} a_{2\rho+1}},$$

then

$$\zeta_0(w_1, \ldots . w_\rho;\ t) = \sum_\alpha g_\alpha(t)\, w_{\rho+\alpha} + \sum_i \frac{\pi(t)}{\pi'(x_i)(t-x_i)} \sum_k{}' \frac{1}{2}\frac{y_i - y_k}{x_i - x_k}, \qquad (\mathrm{A}_0)$$

where

$$\pi(t) = \prod_i (t - x_i),$$

the index i taking, ad libitum, either all the values $1, 2, \ldots . \rho + 1$, *or** ρ *of them, while k takes all the values* $1, 2, \ldots . \rho + 1$ *with the exception of* $k = i$.

This is the extension of theorem (a) mentioned in the introduction. In fact, for $\rho = 1$ and $g_1(x) = 1$, the relation (5) yields for $g_2(x)$ the value

$$g_2(x) = -\tfrac{1}{2}(A_0 x^2 + 2A_1 x + A_2),$$

and (A_0) becomes, if i is allowed to take the value 1 only,

$$\frac{\mathfrak{S}_0'}{\mathfrak{S}_0}(w_1) = \frac{1}{2}\frac{y_1 - y_2}{x_1 - x_2} + w_2.$$

Put $x_2 = x_0$, $y_2 = -y_0$, and let

$$\int_{a_3}^{a_1} \frac{dx}{y} = \omega, \quad \int_{a_3}^{a_1} \frac{g_2(x)\, dx}{y} = \eta,$$

then

$$w_1 = \int_{(x_0 y_0)}^{(x_1 y_1)} \frac{dx}{y} + \omega, \quad w_2 = \int_{(x_0 y_0)}^{(x_1 y_1)} \frac{g_2(x)\, dx}{y} + \eta.$$

On the other hand

$$\frac{\mathfrak{S}_0'}{\mathfrak{S}_0}(u + \omega) = \frac{\mathfrak{S}'}{\mathfrak{S}}(u) + \eta.$$

Thus we obtain

$$\frac{\mathfrak{S}'}{\mathfrak{S}}\left(\int_{(x_0 y_0)}^{(x_1 y_1)} \frac{dx}{y}\right) = \frac{1}{2}\frac{y_1 + y_0}{x_1 - x_0} - \frac{1}{2}\int_{(x_0 y_0)}^{(x_1 y_1)} (A_0 x^2 + 2A_1 x + A_2)\frac{dx}{y},$$

which is precisely theorem (a).

From (A_0) follows as a special case the expression of a sum of ρ normal integrals of the second kind in terms of the fundamental Θ-function, given by Wiltheiss, Journal für Math., Bd. 99, p. 247.

For if we put, in A_0, $x_{\rho+1} = a_{2\rho+1}$ and write

$$\Phi(t) = (t - x_1)(t - x_2) \ldots . (t - x_\rho), \qquad (27)$$

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots . + w_\alpha^{x_\rho a_{2\rho-1}}, \qquad (28)$$

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* Compare the corollary to the lemma of §2.

we have

$$\sum_{\alpha} g_{\alpha}(t) \frac{\partial \log \Theta}{\partial u_{\alpha}} = \sum_{\alpha,\beta} g_{\alpha}(t)\, w_{\rho+\alpha}^{x_{\beta} a_{2\beta-1}} + \sum_{\beta} \frac{\Phi(t)}{\Phi'(x_{\beta})(t-x_{\beta})} \left\{ \frac{1}{2}\, \frac{y_{\beta}}{x_{\beta}-a_{2\rho+1}} + \sum_{\gamma}{}' \, \frac{1}{2}\, \frac{y_{\beta}-y_{\gamma}}{x_{\beta}-x_{\gamma}} \right\}, \tag{29}$$

where the index γ takes the values $1, 2, \ldots . \rho$ with the exception of $\gamma = \beta$.

On the other hand, Wiltheiss' formula, written in our notation, is

$$\sum_{\alpha} g_{\alpha}(t) \frac{\partial \log \Theta}{\partial u_{\alpha}} = \sum_{\alpha,\beta} g_{\alpha}(t)\, w_{\rho+\alpha}^{x_{\beta} a_{2\beta-1}} + \sum_{\beta} \frac{1}{2}\, \frac{y_{\beta}}{x_{\beta}-t} - \sum_{\alpha,\beta} \left(\frac{1}{2}\, \frac{1}{x_{\beta}-t} - \frac{1}{2(x_{\beta}-a_{2\rho+1})} \right) g_{\alpha}(t) \frac{\partial x_{\beta}}{\partial u_{\alpha}}.$$

Both formulæ agree exactly if we make use of the relation

$$\sum_{\alpha} g_{\alpha}(t) \frac{\partial x_{\beta}}{\partial u_{\alpha}} = \frac{\Phi(t)\, y_{\beta}}{\Phi'(x_{\beta})(t-x_{\beta})},$$

which follows from the definition of u_{α}, and of the identity

$$\sum_{\beta} \frac{y_{\beta}}{2(x_{\beta}-t)} + \sum_{\beta} \frac{1}{2}\, \frac{\Phi(t)\, y_{\beta}}{\Phi'(x_{\beta})(t-x_{\beta})^2} = \sum_{\beta} \frac{\Phi(t)}{2\Phi'(x_{\beta})(t-x_{\beta})} \sum_{\gamma}{}' \frac{y_{\beta}-y_{\gamma}}{x_{\beta}-x_{\gamma}}.$$

To prove the latter, observe that both sides are integral functions of t of degree $\rho - 1$ which coincide for the ρ values $t = x_1, x_2, \ldots . x_{\rho}$.

Lastly, we refer to Thomae's researches on integrals of the second kind, Jour. für Math., Bd. 71, p. 213, and Bd. 93, p. 80, where an allied problem is treated.

PART II.

THE SECOND LOGARITHMIC DERIVATIVES.

§5.—*The Analogue of $\wp u$ for $\rho > 1$.*

Before proceeding to the generalization of theorem (b), we must agree what we are to consider as the analogue of $\wp u$ in the hyperelliptic case. Though various ways of generalization are possible, certain results obtained by Brioschi and Wiltheiss, combined with Klein's general theory of hyperelliptic σ-functions, seem to leave no doubt as to the best way.

Brioschi[*] has given for $\rho = 2$ generalizations of the formula

$$\wp u - e\lambda = \frac{\sigma_\lambda^2 u}{\sigma^2 u},$$

which were extended to the general hyperelliptic case by Wiltheiss.[†] In their formulæ the mixed-polar-like expression

$$-\sum_{\alpha,\,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(a_\lambda)\, g_\beta(a_\mu),$$

a_λ, a_μ being two branchpoints, takes the place of $\wp u$.

On the other hand, Wiltheiss[‡] has given a generalization for $\rho = 2$ of the differential equations

$$\begin{aligned} \wp'^2 u &= 4\wp^3 u - g_2 \wp u - g_3, \\ \wp'' u &= 6\wp^2 u - \tfrac{1}{2} g_2. \end{aligned}$$

In his results $\wp u$ is replaced by

$$-\sum_{\alpha,\,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(t)\, g_\beta(t),$$

t being an auxiliary variable, and the same expression occurs incidentally in his researches on the partial differential equations of the hyperelliptic Θ-functions.[||]

These results point to the expression

$$-\sum_{\alpha,\,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(s)\, g_\beta(t),$$

s and t being two auxiliary variables,[§] as the true generalization of $\wp u$.

Accordingly we *define*

$$\left.\begin{aligned} \wp_{\phi\psi}(u_1 \,\ldots\, u_\rho;\, s,\, t) &= -\sum_{\alpha,\,\beta} \frac{\partial^2 \log \sigma_{\phi\psi}(u_1 \,\ldots\, u_\rho)}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(s)\, g_\beta(t) \\ &= -\sum_{\alpha} \frac{\partial\, \zeta_{\phi\psi}(u_1 \,\ldots\, u_\rho;\, t)}{\partial u_\alpha}\, g_\alpha(s). \end{aligned}\right\} \qquad (30)$$

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[*] Rendic. dell' Accademia dei Lincei, 1886_1, p. 199.

[†] Math. Ann., Bd. 31, p. 417.

[‡] Jahresbericht der deutschen Mathematiker-Vereinigung, Bd. I, p. 72.

[||] Math. Ann., Bd. 31, p. 153, and Bd. 33, p. 286.

[§] The introduction of *two* auxiliary variables was suggested to me by Professor Klein.

For $\rho = 1$ and $g_1(x) = 1$ we obtain the four functions

$$\wp u = -\frac{d^2 \log \sigma}{du^2}, \quad \wp(u + \omega_\lambda) = -\frac{d^2 \log \sigma_\lambda}{du^2}, \qquad \lambda = 1, 2, 3.$$

Every $\wp$-function is reducible to the "*fundamental*" *$\wp$-function* whose characteristic is given by (8), and which we shall denote by $\wp_0$, by means of a relation of the form

$$\wp_{\phi\psi}(u_1 + \omega_1, \ldots . u_\rho + \omega_\rho; s, t) = \wp_0(u_1, \ldots . u_\rho; s, t), \tag{31}$$

$\omega_1, \omega_2, \ldots . \omega_\rho$ being a certain system of simultaneous half-periods of the integrals (2).

In the same sense as $\zeta(w_1, \ldots . w_\rho; t)$ in §3, also $\wp(w_1, \ldots . w_\rho; s, t)$ is a *covariant* of $R(x)$, but *of weight* -2; this is easily seen by a repetition of the conclusions of §3.

Concerning the use of the auxiliary variables s and t, an analogous remark applies as in §1; they serve to unite in a simple way theorems concerning the *system of* $\frac{\rho(\rho+1)}{2}$ *functions,*

$$\frac{\partial^2 \log \sigma}{\partial u_\alpha \, \partial u_\beta}.$$

§6.—*An Interpolation Formula.*

For our further developments we need the following *Lemma: Let $G(s, t)$ be an integral symmetric function of s and t, of degree $\rho - 1$ with respect to each of the two variables; further, $x_1, x_2, \ldots . x_\nu$ $\nu \overset{=}{>} \rho + 1$ distinct values, and*

$$\pi(t) = (t - x_1)(t - x_2) \ldots . (t - x_\nu),$$

then

$$G(s, t) = -\sum_{(i,k)} \frac{(x_i - x_k)^2 \, G(x_i, x_k)\, \pi(s)\, \pi(t)}{\pi'(x_i)\, \pi'(x_k)(t - x_i)(t - x_k)(s - x_i)(s - x_k)}, \tag{32}$$

the summation extending over the $\frac{\nu(\nu-1)}{2}$ two-combinations of the numbers $1, 2, \ldots . \nu$.

Proof: The function $G(x_i, t)$ is an integral function of t of degree $\rho - 1$; the function

$$\pi_i(t) = \frac{\pi(t)}{t - x_i}$$

is at least of degree ρ; hence by decomposition into partial fractions

$$\frac{G(x_i, t)}{\pi_i(t)} = \sum_k{}' \frac{G(x_i, x_k)}{\pi_i'(x_k)(t - x_k)},$$

the index k taking the values $1, 2, \ldots . \nu$ with the exception of $k = i$. But

$$\pi_i'(x_k) = \frac{\pi'(x_k)}{x_k - x_i};$$

hence

$$G(x_i, t) = \sum_k{}' \frac{(x_k - x_i)\, G(x_i, x_k)\, \pi(t)}{\pi'(x_k)(t - x_i)(t - x_k)}.$$

Next consider the function $G(s, t)$ as a function of s alone; again, by decomposition into partial fractions,

$$\frac{G(s, t)}{\pi(s)} = \sum_i \frac{G(x_i, t)}{\pi'(x_i)(s - x_i)},$$

the index i running from 1 to ν.

Substitute the above value of $G(x_i, t)$, observe that generally

$$\sum_i \sum_k{}' c_{ik} = \sum_{(i,k)} (c_{ik} + c_{ki}),$$

and make use of our hypothesis

$$(x_i, x_k) = G(x_k, x_i);$$

thus we obtain

$$\frac{G(s, t)}{\pi(s)} = \sum_{(i,k)} \frac{(x_k - x_i)\, G(x_i, x_k)\, \pi(t)}{\pi'(x_i)\, \pi'(x_k)(t - x_i)(t - x_k)} \left[\frac{1}{s - x_i} - \frac{1}{s - x_k}\right],$$

which leads immediately to (32).

§7.—*Generalization of Theorem* (b).

As in §2, we *replace the independent variables* u_α *of* $\wp(u_1, \ldots . u_\rho; s, t)$ *by sums of* ν *integrals of the first kind,*

$$w_\alpha = w_\alpha^{x_1\xi_1} + w_\alpha^{x_2\xi_2} + \ldots . + w_\alpha^{x_\nu\xi_\nu}, \tag{10}$$

supposing, however, $\nu \overset{=}{>} \rho + 1$.

$\wp(w_1, \ldots . w_\rho; s, t)$ is an integral symmetric function of s and t, of degree $\rho - 1$ in each of the two variables. Therefore the Lemma of last section is applicable, by which the computation of $\wp(w_1, \ldots . w_\rho;\, s,\, t)$ is reduced to the computation of the $\frac{\nu(\nu-1)}{2}$ special values

$$\wp(w_1, \ldots . w_\rho;\, x_i,\, x_k) \qquad (\varkappa \neq i).$$

But from (27) follows

$$\wp(w_1, \ldots . w_\rho;\, x_i,\, x_k) = -\sum_{\alpha,\,\beta} \frac{\partial^2 \log \sigma}{\partial w_\alpha\, \partial w_\beta}\, g_\alpha(x_i)\, g_\beta(x_k) = -\, y_i y_k \frac{\partial^2 \log \sigma}{\partial x_i\, \partial x_k}. \tag{33}$$

For our further developments we make again the same restricting assumptions as in §4, viz. we confine ourselves to the fundamental $\wp$-function $\wp_0$ and choose

$$w_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots . + w_\alpha^{x_{\rho+1} a_{2\rho+1}}. \tag{25}$$

Differentiating (26) with respect to x_k, we get

$$\left.\begin{aligned} y_i y_k \frac{\partial^2 \log \sigma_0}{\partial x_i\, \partial x_k} &= y_k \frac{\partial}{\partial x_k} \frac{y_i - y_k}{x_i - x_k} + \sum_\alpha g_\alpha(x_i)\, g_{\rho+\alpha}(x_k), \\ \text{or, according to (5),} \qquad &= \frac{y_i y_k - F(x_i,\, x_k)}{2\,(x_i - x_k)^2} \end{aligned}\right\} \tag{34}$$

Thus we obtain the

Theorem: If

$$w_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots . + w_\alpha^{x_{\rho+1} a_{2\rho+1}},$$

then

$$\wp_0(w_1,\, w_2, \ldots . w_\rho;\, x_i,\, x_k) = \frac{-\,y_i y_k + F(x_i,\, x_k)}{2\,(x_i - x_k)^2} \tag{B}$$

and

$$\wp_0(w_1,\, w_2, \ldots . w_\rho;\, s,\, t) = \sum_{(i,\,k)} \frac{\bigl(y_i y_k - F(x_i,\, x_k)\bigr)\, \pi(s)\, \pi(t)}{2\pi'(x_i)\, \pi'(x_k)(s - x_i)(s - x_k)(t - x_i)(t - x_k)}, \tag{B'}$$

the summation extending over the $\frac{\rho(\rho+1)}{2}$ *two-combinations of the numbers* $1, 2, \ldots . \rho + 1$. This is an extension of theorem (b). In fact, for $\rho = 1$ and $g_1(x) = 1$, (B) reduces to

$$\wp_0(w_1) = \frac{-\,y_1 y_2 + F(x_1,\, x_2)}{2\,(x_1 - x_2)^2},$$

where

$$w_1 = w_1^{x_1 a_1} + w_1^{x_2 a_3}.$$

But if we put again, as in §4, $x_2 = x_0$, $y_2 = -\,y_0$ and

$$\omega = \int_{a_3}^{a_1} \frac{dx}{y},$$

we have
$$\wp_0(w_1) = \wp_0(w_1^{x_1 x_0} + \omega) = \wp(w_1^{x_1 x_0}),$$
whence
$$\wp(w_1^{x_1 x_0}) = \frac{y_0 y_1 + F(x_0, x_1)}{2(x_0 - x_1)^2}. \qquad \text{Q. E. D.}$$

Second proof of (B): We add another proof of (B) which does not presuppose (26), but starts directly from the well-known expression of the quotient of two products of Θ- (or σ-) functions in terms of integrals of the third kind. Let

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \dots\dots + w_\alpha^{x_\rho a_{2\rho-1}},$$
$$u'_\alpha = w_\alpha^{\xi_1 a_1} + w_\alpha^{\xi_2 a_3} + \dots\dots + w_\alpha^{\xi_\rho a_{2\rho-1}},$$

then*
$$\log \frac{\sigma_0((w_\alpha^{x a_{2\rho+1}} - u_\alpha))\,\sigma_0((w_\alpha^{\xi a_{2\rho+1}} - u'_\alpha))}{\sigma_0((w_\alpha^{x a_{2\rho+1}} - u'_\alpha))\,\sigma_0((w_\alpha^{\xi a_{2\rho+1}} - u_\alpha))} = \sum_\beta Q_{x\xi}^{x_\beta \xi_\beta}. \tag{35}$$

Hence, differentiating with respect to x and x_β,

$$\frac{\partial^2 \log \sigma_0((w_\alpha^{x a_{2\rho+1}} - u_\alpha))}{\partial x\, \partial x_\beta} = \frac{\partial^2 Q_{x\xi}^{x_\beta \xi_\beta}}{\partial x\, \partial x_\beta} = \frac{y y_\beta + F(x, x_\beta)}{2(x - x_\beta)^2 y y_\beta}.$$

Replacing the point $x = (x, y)$ by $\overline{x}_{\rho+1} = (x_{\rho+1}, -y_{\rho+1})$, we get

$$\wp_0(w_1, \dots\dots w_\rho;\, x_\beta, \ddot{x}_{\rho+1}) = \frac{-y_\beta y_{\rho+1} + F(x_\beta, x_{\rho+1})}{2(x_\beta - x_{\rho+1})^2},$$

$w_1, \dots\dots w_\rho$ having the same meaning as in (B). Hence follows by a permutation of the letters $x_1, x_2, \dots\dots x_{\rho+1}$, equation (B), which thus appears as an immediate consequence of (35).

Other form of (B′): Formula (B′) may be thrown into another shape which is better adapted for applications. Add and subtract on the right-hand side

$$\sum_i \frac{R(x_i)\,\pi(s)\,\pi(t)}{4\pi'^2(x_i)(s - x_i)^2(t - x_i)^2}$$

and observe that

$$R(x_i) = y_i^2 = F(x_i, x_i)$$

and

$$\sum_i \sum_k c_{ik} = \sum_i c_{ii} + 2\sum_{(i,k)} c_{ik},$$

provided $c_{ik} = c_{ki}$. The right-hand side of (B′) becomes

$$\pi(s)\,\pi(t)\Big[\sum_i \frac{y_i}{2(s - x_i)(t - x_i)\,\pi'(x_i)}\Big]^2$$
$$-\sum_i \sum_k \frac{F(x_i, x_k)\,\pi(s)\,\pi(t)}{4\pi'(x_i)\,\pi'(x_k)(s - x_i)(s - x_k)(t - x_i)(t - x_k)}$$

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* Klein, Math. Ann., Bd. 32, p. 379.

But by decomposition into partial fractions

$$\sum_k \frac{F(x_i, x_k)}{(s-x_k)(t-x_k)\pi'(x_k)} = \frac{1}{t-s}\left[\frac{F(x_i, s)}{\pi(s)} - \frac{F(x_i, t)}{\pi(t)}\right],$$

likewise

$$\sum_i \frac{F(x_i, s)}{\pi'(x_i)(s-x_i)(t-x_i)} = \frac{1}{t-s}\left[\frac{F(s, s)}{\pi(s)} - \frac{F(s, t)}{\pi(t)}\right],$$

and remembering that

$$F(s, s) = R(s), \quad F(t, t) = R(t),$$

we finally obtain the

Theorem: If

$$w_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots . \, w_\alpha^{x_{\rho+1} a_{2\rho+1}},$$

then

$$\left.\begin{aligned} \wp_0(w_1, w_2, \ldots . \, w_\rho;\, s, t) = \pi(s)\pi(t)&\left[\sum_i \frac{y_i}{2(s-x_i)(t-x_i)\pi'(x_i)}\right]^2 \\ &+ \frac{F(s, t)}{2(t-s)^2} - \frac{R(s)\pi(t)}{4(t-s)^2\pi(s)} - \frac{R(t)\pi(s)}{4(s-t)^2\pi(t)}. \end{aligned}\right\} \quad (B'')$$

The special case $s = t$: If we wish to apply (B″) to the case $s = t$ we must first perform the division by $(t-s)^2$; to this effect, introduce homogeneous variables, multiply by $s_2^{\rho-1} t_2^{\rho-1}$ and use symbolical notation

$$R(t) = a_t^{2\rho+2}, \quad F(s, t) = a_t^{\rho+1} a_s^{\rho+1}, \quad \pi(t) = \pi_t^{\rho+1} = p_t^{\rho+1}.$$

The last three terms of (B″) become

$$-\frac{(a_s^{\rho+1}\pi_t^{\rho+1} - a_t^{\rho+1}\pi_s^{\rho+1})(a_s^{\rho+1}p_t^{\rho+1} - a_t^{\rho+1}p_s^{\rho+1}}{4(st)^2\pi(t)\pi(s)};$$

now the division can be immediately performed, and if we put afterwards $s = t$, we get

$$\wp_0(w_1, \ldots . \, w_\rho;\, t, t) = \pi(t)^2\left[\sum_i \frac{y_i}{2(tx_i)^2 \prod_k{}'(x_i x_k)}\right]^2 - \frac{(\rho+1)^2(a\pi)(ap)\, a_t^{2\rho}\pi_t^\rho p_t^\rho}{4\pi(t)^2}. \quad (36)$$

§8.—*Generalization of* (c) *and* (d).

Giving special values either to the upper limits $x_1, x_2, \ldots . \, x_{\rho+1}$ or to the auxiliary variables s and t, we obtain a series of consequences of our last theorem which are extensions of well-known properties of the elliptic function $\wp u$.

In the first place, put in (B′) $t = x_{\rho+1}$ and observe that

$$\left[\frac{\pi(t)}{(t-x_i)(t-x_k)}\right]_{t=x_{\rho+1}} = \begin{cases} 0 \text{ if } i \text{ and } k \ddagger \rho+1, \\ \dfrac{\pi'(x_{\rho+1})}{x_{\rho+1}-x_i} \text{ if } k=\rho+1; \end{cases}$$

(B′) reduces to

$$\wp_0(w_1, w_2, \ldots. w_\rho;\ s, x_{\rho+1}) = \sum_\alpha \frac{\Phi(s)}{(s-x_\alpha)\,\Phi'(x_\alpha)}\ \frac{F(x_\alpha, x_{\rho+1}) - y_\alpha y_{\rho+1}}{2(x_\alpha - x_{\rho+1})^2},$$

where, as in §4,

$$\Phi(s) = (s-x_1)(s-x_2)\ldots.(s-x_\rho). \tag{27}$$

Now give $x_{\rho+1}$ the special value $x_{\rho+1} = a_{2\rho+1}$ and denote, as in §4,

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots. + w_\alpha^{x_\rho a_{2\rho-1}}; \tag{28}$$

then

$$\wp_0(u_1, u_2, \ldots. u_\rho;\ s, a_{2\rho+1}) = \sum_\alpha \frac{\Phi(s)}{\Phi'(x_\alpha)(s-x_\alpha)}\ \frac{F(x_\alpha, a_{2\rho+1})}{2(x_\alpha - a_{2\rho+1})^2}.$$

But a being any branchpoint, we have

$$F(a, a) = R(a) = 0,$$
$$\left(\frac{dF(x, a)}{dx}\right)_{x=a} = \frac{1}{2}R'(a),$$

therefore

$$\frac{F(x, a) - \frac{1}{2}R'(a)(x-a)}{2(x-a)^2} = G(x)$$

is an *integral* function of x of degree $\rho - 1$, hence

$$\sum_\alpha \frac{\Phi(s)\,G(x_\alpha)}{\Phi'(x_\alpha)(s-x_\alpha)} = G(s).$$

On the other hand, decomposition into partial fractions of $\dfrac{1}{\Phi(s)(s-a)}$ gives

$$\sum_\alpha \frac{\Phi(s)}{\Phi'(x_\alpha)(s-x_\alpha)(x_\alpha-a)} = \frac{1}{s-a} - \frac{\Phi(s)}{\Phi(a)(s-a)}.$$

These transformations lead to the following

Theorem: If $\quad u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \ldots. w_\alpha^{x_\rho a_{2\rho-1}},$

then

$$\wp_0(u_1, u_2, \ldots. u_\rho;\ s, a_{2\rho+1})\ \frac{F(s, a_{2\rho+1})}{2(s-a_{2\rho+1})^2} - \frac{1}{4}\ \frac{R'(a_{2\rho+1})\,\Phi(s)}{\Phi(a_{2\rho+1})(s-a_{2\rho+1})}. \tag{C}$$

Special case $\rho = 1$, $g_1(x) = 1$:

$$\wp_0(w_1^{x_1 a_1}) = \frac{F(s, a_3)}{2(s-a_3)^2} - \frac{1}{4}\,\frac{R'(a_3)}{s-a_3} + \frac{1}{4}\,\frac{R'(a_3)}{x-a_3}.$$

But

$$F(s, a_3) = \frac{1}{2} R'(a_3)(s-a_3) + \frac{1}{12} R''(a_3)(s-a_3)^2,$$

hence

$$\wp_0(w_1^{x_1 a_1}) = \frac{1}{24} R''(a_3) + \frac{1}{4}\,\frac{R'(a_3)}{x_1 - a_3} = \wp(w_1^{x_1 a_3}),$$

(C) is therefore indeed an extension of the theorem (c) of the introduction.

If $a_{2\rho+1}$ is made infinitely great, (C) changes into a formula given by Wiltheiss. In fact, choose $g_\alpha(x) = \frac{x^{\alpha-1}}{2}$ and put $a_{2\rho+1} = \frac{1}{\varepsilon}$; multiply (C) by $\varepsilon^{\rho-1}$ and pass to the limit $\varepsilon = 0$, while at the same time $\lim \frac{A_0}{\varepsilon}$ remains finite, say

$$\lim \frac{A_0}{\varepsilon} = (2\rho + 2) A_1',$$

so that

$$\lim R(x) = \binom{2\rho+2}{1} A_1' x^{2\rho+1} + \binom{2\rho+2}{2} A_2' x^{2\rho} + \dots\dots A_{2\rho+2}'.$$

The result of the limiting process is

$$\sum_\alpha \frac{\partial^2 \log \Theta}{\partial u_\alpha \, \partial u_\rho} s^{\alpha-1} = -2F(s) + (2\rho+2) A_1' \Phi(s), \tag{C'}$$

where

$$F(s) = \binom{\rho+1}{1} A_1' s^\rho + \binom{\rho+1}{2} A_2' s^{\rho-1} + \dots\dots + A_{\rho+1}.$$

This is the formula given by Wiltheiss, Math. Ann., Bd. 31, p. 414, (8).

Theorem (C) contains a *solution of the inversion problem:*

$$u_\alpha = w_\alpha^{x_1 a_1} + w_\alpha^{x_2 a_3} + \dots\dots + w_\alpha^{x_\rho a_{2\rho-1}}.$$

For the equation of degree ρ, $\Phi(s) = 0$, which determines the ρ values of $x_1, x_2, \dots\dots x_\rho$, is, according to (C),

$$(s - a_{2\rho+1})\, \wp_0(u_1, u_2, \dots\dots u_\rho;\ s, a_{2\rho+1}) - \frac{F(s, a_{2\rho+1})}{2(s - a_{2\rho+1})} = 0. \tag{C''}$$

(Remember that $F(s, a_{2\rho+1})$ is divisible by $s - a_{2\rho+1}$).

The values $x_1, x_2, \dots\dots x_\rho$ being found, for every x_δ the corresponding value y_δ is given by the equation

$$\sum_{\alpha,\beta,\gamma} \frac{\partial^3 \log \sigma_0}{\partial u_\alpha \, \partial u_\beta \, \partial u_\gamma} g_\alpha(a_{2\rho+1})\, g_\beta(a_{2\rho+1})\, g_\gamma(x_\delta) = \frac{1}{4}\,\frac{R'(a_{2\rho+1})}{(a_{2\rho+1} - x_\delta)^2} y_\delta, \tag{D}$$

an extension of (d) which is derived from (C) by differentiating with respect to x_δ and putting afterwards $s = a_{2\rho+1}$.

§9.—*Generalization of* (e).

Put in (B″) the $\rho + 1$ upper limits $x_1, x_2, \ldots . x_{\rho+1}$ equal to $\rho + 1$ distinct branchpoints, say

$$b_1, b_2, \ldots . b_{\rho+1},$$

and let

$$\left.\begin{aligned} \phi(x) &= (x - b_1)(x - b_2) \ldots . (x - b_{\rho+1}), \\ \psi(x) &= \frac{R(x)}{\phi(x)}, \end{aligned}\right\} \tag{37}$$

and

$$\omega_\alpha = w_\alpha^{b_1 a_1} + w_\alpha^{b_2 a_3} + \ldots . w_\alpha^{b_{\rho+1} a_{2\rho+1}}, \tag{38}$$

(B″) becomes

$$\wp_0(\omega_1, \omega_2, \ldots . \omega_\rho ; s, t) = \frac{2F(s, t) - \phi(s)\psi(t) - \phi(t)\psi(s)}{4(t-s)^2}. \tag{E}$$

Introduce homogeneous variables* and use symbolical notation,

$$R(t) = a_t^{2\rho+2}, \ \phi(t) = \phi_t^{\rho+1}, \ \psi(t) = \psi_t^{\rho+1},$$

then

$$F(s, t) = a_s^{\rho+1} a_t^{\rho+1} = \frac{1}{\binom{2\rho+2}{\rho+1}} \left\{ \phi_s^{\rho+1}\psi_t^{\rho+1} + \binom{\rho+1}{1}^2 \phi_s^\rho \psi_t^\rho \phi_t \psi_s \right.$$
$$\left. + \binom{\rho+1}{2}^2 \phi_s^{\rho-1}\psi_t^{\rho-1}\phi_t^2\psi_s^2 + \ldots . + \phi_t^{\rho+1}\psi_s^{\rho+1} \right\},$$

and if we make use of the identity

$$1 + \binom{\rho+1}{1}^2 + \binom{\rho+1}{2}^2 + \ldots . + \binom{\rho+1}{1}^2 + 1 = \binom{2\rho+2}{\rho+1}$$

we obtain, after a few simplifications, the *result:*

$$\wp_0(\omega_1, \omega_2, \ldots . \omega_\rho ; s, t) = -\frac{(\phi\psi)^2 \sum_\alpha \binom{\rho+1}{\alpha}^2 S_{\alpha-1} S_{\rho-\alpha}}{4\binom{2\rho+2}{\rho+1}}, \tag{E′}$$

where $S_i = \phi_s^i \psi_t^i + \phi_s^{i-1}\phi_t \psi_s \psi_t^{i-1} + \ldots . + \phi_s \phi_t^{i-1} \psi_s^{i-1} \psi_t + \phi_t^i \psi_s^i$

and $S_0 = 1$.

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* See the agreement concerning notation at the end of §1.

For $\rho = 1$, (E′) gives

$$\wp_0(\omega_1) = -\frac{1}{6}(\phi\psi)^2,$$

where $$\omega_1 = w_1^{b_1 a_1} + w_1^{b_2 a_3};$$

but $$\wp_0(\omega_1) = \wp_0(w_1^{b_1 b_2} + w_1^{a_1 a_3}) = \wp(w_1^{b_1 b_2});$$

hence $\wp\left(\int_{b_2}^{b_1} \frac{dx}{y}\right) = -\frac{1}{6}(\phi\psi)^2$, which is theorem (e) of the introduction.

For $\rho = 2$, (E′) gives

$$\wp_0(\omega_1, \omega_2; s, t) = -\frac{9}{40}\{(\phi\psi)^2 \phi_s \psi_t + (\phi\psi)^2 \phi_t \psi_s\}. \tag{39}$$

If the auxiliary variables s and t are replaced by two of the branchpoints b, say b_i, b_k, (E) becomes

$$\wp_0(\omega_1, \omega_2, \ldots . \omega_\rho; b_i, b_k) = \frac{F(b_i, b_k)}{2(b_i - b_k)^2}. \tag{40}$$

§10.—*Generalization of* $\wp u - e_\lambda = \frac{\sigma_\lambda^2 u}{\sigma^2 u}$.

A first extension of this theorem is immediately derived from (C) by replacing s by a branchpoint different from $a_{2\rho+1}$, say

$$s = a_\lambda, \quad \lambda \ddagger 2\rho + 1.^*$$

We obtain

$$\wp(u_1, \ldots . u_\rho; a_\lambda, a_{2\rho+1}) - \frac{F(a_\lambda, a_{2\rho+1})}{2(a_\lambda - a_{2\rho+1})^2} = -\frac{1}{4}\frac{R'(a_{2\rho+1})\Phi(a_\lambda)}{(a_\lambda - a_{2\rho+1})\Phi(a_{2\rho+1})}. \tag{41}$$

But if we use Weierstrass'† index notation

$$\frac{\Phi(a_\lambda)}{\Phi(a_{2\rho+1})} = \frac{\sqrt{(-1)^\lambda R'(a_\lambda)}}{\sqrt{-R'(a_{2\rho+1})}} \frac{\Theta^2(u_1, \ldots . u_\rho)\lambda}{\Theta^2(u_1, \ldots . u_\rho)}; \tag{42}$$

hence if we write $$\frac{F(a_i, a_k)}{2(a_i - a_k)^2} = e_{ik}, \tag{43}$$

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*For $\lambda = 2\rho + 1$ we get

$$\wp_0(u_1, \ldots . u_\rho; a_{2\rho+1}, a_{2\rho+1}) = \frac{\rho}{8(2\rho+1)} R''(a_{2\rho}+1) - \tfrac{1}{4}R'(a_{2\rho}+1)\frac{\Phi'(a_{2\rho}+1)}{\Phi(a_{2\rho}+1)}.$$

† Weierstrass, Lectures on Hyperelliptic Functions, Solution of the inversion problem.

we obtain the *result:*

$$\wp_0(u_1,\ldots\, u_\rho;\, a_\lambda,\, a_{2\rho+1}) = e_{\lambda,\,2\rho+1}$$
$$= \frac{1}{4}\,\frac{\sqrt{(-1)^{\lambda+1}R'(a_\lambda)\,R'(a_{2\rho+1})}}{a_\lambda - a_{2\rho+1}}\,\frac{\Theta^2(u_1,\ldots\, u_\rho)_\lambda}{\Theta^2(u_1,\ldots\, u_\rho)}. \qquad \text{(F)}$$

For $\rho = 2$ and $a_{2\rho+1} = \infty$, this result was first proved by Brioschi, Rend. della Acc. dei Lincei, 1886_1, p. 199; the general formula (F) was given by Wiltheiss, Math. Ann., Bd. 31, p. 417.

An analogous formula for Θ-functions with two indices is obtained by putting in (B″)

$$x_{\rho+1} = a_{2\rho+1}, \quad s = a_\lambda, \quad t = a_\mu .$$

(B″) becomes

$$\wp_0(u_1,\ldots\, u_\rho;\, a_\lambda,\, a_\mu) - \frac{F(a_\lambda,\, a_\mu)}{2\,(a_\lambda - a_\mu)^2}$$
$$= (a_\lambda - a_{2\rho+1})(a_\mu - a_{2\rho+1})\,\Phi(a_\lambda)\,\Phi(a_\mu)\left\{\sum_\alpha \frac{y_\alpha}{2\,(x_\alpha - a_\lambda)(x_\alpha - a_\mu)(x_\alpha - a_{2\rho+1})\,\Phi'(x_\alpha)}\right\}^2. \qquad (44)$$

But according to Weierstrass,

$$\frac{(a_{2\rho+1} - a_\lambda)(a_\mu - a_\lambda)(a_\mu - a_{2\rho+1})\,\Phi^2(a_{2\rho+1})}{R'(a_{2\rho+1})}\left\{\sum_\alpha \frac{y_\alpha}{(x_\alpha - a_\lambda)(x_\alpha - a_\mu)(x_\alpha - a_{2\rho+1})\,\Phi'(x_\alpha)}\right\}^2$$
$$= \pm\,\frac{\Theta^2(u_1,\ldots\, u_\rho)_{\lambda\mu}\,\Theta^2(u_1,\ldots\, u_\rho)}{\Theta^2(u_1,\ldots\, u_\rho)_\lambda\,\Theta^2(u_1,\ldots\, u_\rho)_\mu}. \qquad (45)$$

Hence we obtain the *result:*

$$\wp_0(u_1,\, u_2,\ldots\, u_\rho;\, a_\lambda a_\mu) - e_{\lambda\mu}$$
$$= \frac{1}{4}\,\frac{\sqrt{(-1)^\lambda R'(a_\lambda)}\sqrt{(-1)^\mu R'(a_\mu)}}{a_\lambda - a_\mu}\,\frac{\Theta^2(u_1,\, u_2,\ldots\, u_\rho)_{\lambda\mu}}{\Theta^2(u_1,\, u_2,\ldots\, u_\rho)}. \qquad \text{(G)}$$

This agrees exactly with the result given by Brioschi l. c. for $\rho = 2$, $a_{2\rho+1} = \infty$, if we make use of the identity

$$\frac{F(a_\lambda,\, a_\mu)}{2\,(a_\lambda - a_\mu)^2} = \frac{1}{4}\,\frac{R'(a_\lambda)}{a_\mu - a_\lambda} - \sum_\alpha g_{\rho+\alpha}(a_\lambda)\,g_\alpha(a_\mu) = \frac{1}{4}\,\frac{R'(a_\mu)}{a_\lambda - a_\mu} - \sum_\alpha g_{\rho+\alpha}(a_\mu)\,g_\alpha(a_\lambda),$$

which follows from (5) for $\xi = a_\lambda$, $x = a_\mu$.

The symmetry with respect to the branchpoints is better seen, and at the same time the irrational coefficients are avoided, if Klein's *σ-functions and alge-*

braic characteristics are introduced instead of Weierstrass' Θ-functions and index notation. This may be done either by means of the relations which connect the Θ- and σ-functions (see Klein, Math. Ann., Bd. 32, p. 376, and Schröder, Ueber den Zusammenhang der hyperelliptischen σ- und ϑ-Functionen, Diss. Göttingen, 1890), or else by using Klein's theorem* (15). The *result* is as follows:

Let as before

$$\left.\begin{aligned} \phi_0(x) &= \text{const.}\,(x-a_1)(x-a_3)\,\ldots\,(x-a_{2\rho+1}), \\ \psi_0(x) &= \text{const.}\,(x-a_0)(x-a_2)\,\ldots\,(x-a_{2\rho}) \end{aligned}\right\} \tag{8}$$

and

$$\sigma_0 = \sigma_{\phi_0\psi_0}.$$

Two essentially different cases have to be distinguished.

I. *Case*: a_λ, a_μ belong the one to ϕ_0, the other to ψ_0, say $\lambda = 2i-1$, $\mu = 2k$. Put

$$\left.\begin{aligned} \phi(x) &= \frac{x-a_{2k}}{x-a_{2i-1}}\,\phi_0(x), \\ \psi(x) &= \frac{x-a_{2i-1}}{x-a_{2k}}\,\psi_0(x), \end{aligned}\right\} \tag{46}$$

then

$$\wp_0(u_1, u_2, \ldots\, u_\rho;\, a_{2i-1}, a_{2k}) - e_{2i-1,\,2k} = \frac{1}{4}\,\phi'(a_{2k})\,\psi'(a_{2i-1})\,\frac{\sigma^2_{\phi\psi}(u_1, u_2, \ldots\, u_\rho)}{\sigma^2_0(u_1, u_2, \ldots\, u_\rho)}. \tag{H}$$

II. *Case*: a_λ, a_μ belong both to the same of the two factors ϕ_0, ψ_0, say for instance to ϕ_0, $\lambda = 2i-1$, $\mu = 2k-1$. Put

$$\begin{aligned} p(x) &= \frac{\phi_0(x)}{(x-a_{2i-1})(x-a_{2k-1})}, \\ \chi(x) &= \psi_0(x)(x-a_{2i-1})(x-a_{2k-1}), \end{aligned} \tag{47}$$

then

$$\begin{aligned} &\wp_0(u_1, u_2, \ldots\, u_\rho;\, a_{2i-1}, a_{2k-1}) - e_{2i-1,\,2k-1} \\ &\qquad = \frac{1}{16}\,\psi_0(a_{2i-1})\,\psi_0(a_{2k-1})\,\frac{\sigma^2_{p\chi}(u_1, u_2, \ldots\, u_\rho)}{\sigma^2_0(u_1, u_2, \ldots\, u_\rho)}. \end{aligned} \tag{I}$$

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* For the value of the constant factor $c^{(\mu)}_\nu$ see Burkhardt, Math. Ann., Bd. 32, p. 418.

Verification for $\rho = 1$: Choose $g_1(x) = 1$ and the constant factor in $\phi_0(x)$ equal 1; then

$$\begin{array}{ll} \phi_0(x) = (x - a_1)(x - a_3), & \psi_0(x) = A_0(x - a_0)(x - a_2), \\ p(x) = 1, & \chi(x) = R(x). \end{array}$$

Further choose, in order to fix the ideas, in (46),

$$a_{2i-1} = a_3, \quad a_{2k} = a_0,$$

so that $\phi(x) = (x - a_0)(x - a_1), \quad \psi(x) = A_0(x - a_2)(x - a_3)$;

(H) and (I) become

$$\wp_0(u) - e_{03} = \frac{1}{4} A_0 (a_0 - a_1)(a_3 - a_2) \frac{\sigma^2_{\phi\psi}(u)}{\sigma^2_0(u)}, \tag{h}$$

$$\wp_0(u) - e_{13} = \frac{1}{16} A_0^2 (a_1 - a_0)(a_1 - a_2)(a_3 - a_0)(a_3 - a_2) \frac{\sigma^2(u)}{\sigma^2_0(u)}. \tag{i}$$

But if we express $F(a_i, a_k)$ in terms of the roots we obtain*

$$e_{ik} = \frac{F(a_i, a_k)}{2(a_i - a_k)^2} = -\frac{A_0}{12}\left[(a_i - a_l)(a_k - a_m) + (a_i - a_m)(a_k - a_l)\right],$$

i, k, l, m denoting the four numbers 0, 1, 2, 3 taken in any order; therefore

$$e_{ik} = e_{lm}$$

and

$$e_{ik} - e_{il} = -\frac{A_0}{4}(a_i - a_m)(a_k - a_l).$$

Hence if we designate

$$e_{13} = e_{02} = e_\lambda; \quad e_{10} = e_{23} = e_\mu; \quad e_{12} = e_{03} = e_\nu,$$

$$\omega_\lambda = \int_{a_1}^{a_3} \frac{dx}{y},$$

and accordingly $\sigma_0 = \sigma_\lambda, \quad \sigma_{\phi\psi} = \sigma_\mu,$

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* Compare Weierstrass, Lectures on Elliptic Functions, and Klein, Elliptische Modulfunctionen, I, p. 17.

(h) and (i) become

$$\wp(u+\omega_\lambda)-e_\nu=(e_\lambda-e_\nu)\frac{\sigma_\mu^2(u)}{\sigma_\lambda^2(u)}, \tag{h'}$$

$$\wp(u+\omega_\lambda)-e_\lambda=(e_\lambda-e_\mu)(e_\lambda-e_\nu)\frac{\sigma^2(u)}{\sigma_\lambda^2(u)}, \tag{i'}$$

and if we apply the formula*

$$\wp(u+\omega_\lambda)-e_\lambda=\frac{(e_\lambda-e_\mu)(e_\lambda-e_\nu)}{\wp u-e_\lambda}, \tag{48}$$

we obtain the well-known equations

$$\frac{\wp(u)-e_\mu}{\wp(u)-e_\lambda}=\frac{\sigma_\mu^2(u)}{\sigma_\lambda^2(u)}, \tag{h''}$$

$$\frac{1}{\wp(u)-e_\lambda}=\frac{\sigma^2(u)}{\sigma_\lambda^2(u)}. \tag{i''}$$

An *extension of* (48) may be deduced from (H) as follows: Interchange the two branchpoints a_{2i-1} and a_{2k}, the result is

$$\wp_{\phi\psi}(u_1, \dots . u_\rho;\ a_{2k}, a_{2i-1})-e_{2i-1,\,2k}=\frac{1}{4}\,\phi_0'(a_{2i-1})\,\psi_0'(a_{2k})\,\frac{\sigma_0^2(u_1, \dots . u_\rho)}{\sigma_{\phi\psi}^2(u_1, \dots . u_\rho)}. \tag{49}$$

Multiply (H) and (49) and observe that (31),

$$\wp_{\phi\psi}(u_1, \dots, u_\rho;\ s,\ t)=\wp_0(u_1+\varpi_1, \dots . u_\rho+\varpi_\rho;\ s,\ t) \tag{50}$$

if

$$\varpi_\alpha=\int_{a_{2i-1}}^{a_{2k}}\frac{g_\alpha(x)\,dx}{y}, \tag{51}$$

therefore

$$\wp_0(u_1+\varpi_1, \dots . u_\rho+\varpi_\rho;\ a_{2i-1}, a_{2k})-e_{2i-1,\,2k}$$
$$=\frac{1}{16}\,\frac{\phi'(a_{2k})\,\psi'(a_{2i-1})\,\phi_0'(a_{2i-1})\,\psi_0'(a_{2k})}{\wp_0(u_1, \dots . u_\rho;\ a_{2i-1}, a_{2k})}. \tag{K}$$

In concluding, I may remark that most of our developments—in particular the fundamental equations (A), (B), (B″)—are independent of the special assumption which we made concerning the commutative integral of the third kind. We might just as well have chosen, instead of Klein's integral Q, the *most general commutative integral of the third kind*, and accordingly, instead of Klein's σ-functions, the most general σ-functions. We would only have to replace in all our formulæ the $\rho+1^{\text{st}}$ polar $F(x, \xi)$ by the most general integral symmetric function of x and ξ, $F(x, \xi)$ of degree $\rho + 1$ with respect to each of the two variables satisfying the two conditions*

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*Schwarz's Formelsammlung, Art. 19, (5).

$$F(\xi, \xi) = R(\xi),$$
$$\left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} = \frac{1}{2} R'(\xi).$$

Our results would thus gain in generality, but they would lose their covariant character.

FREIBURG i. B., *August*, 1894.

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* Weierstrass, Lectures on Hyperelliptic Functions; Klein, Math. Ann., Bd. 27, p. 441, and Lectures on Hyperelliptic Functions, 1887–88.